Comment. Math.
Univ. St. Pauli
XXVI—1, 1977

On the multiplier extension of commutative H^*-algebras

by

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(Received Jaunary 10, 1977)

1. Introduction

This paper is closely related to [4], [2] and Chapter 2 in [5], and extends some of the results of [9] and [10], but is mainly based on the first section of [11], where the multiplier extension of admissible vector modules was described to obtain a common algebraic view of Schwartz distributions and Mikusiński operators.

The multiplier extension of an admissible vector module is an admissible vector module containing the scalar field and the corresponding module. Its elements are called quotient multipliers since the multiplier extension of an admissible algebra (i.e., of a commutative algebra without proper annihilators being considered as an admissible vector module over itself) is a generalization of its classical quotient algebra.

In this paper, the multiplier extension $\mathfrak{M}$ of a commutative H^*-algebra $\mathscr{A}$ is investigated mainly from an algebraic point of view, but some topological considerations are also involved. In the light of the Gelfand representation theory, the main result of this paper is that the Gelfand mapping of $\mathscr{A}$ can be extended to an isomorphism of $\mathfrak{M}$ onto the algebra of all continuous complex functions on the maximal ideal space of $\mathscr{A}$ which maps the set of all continuous elements of $\mathfrak{M}$ onto the set of all bounded elements of the corresponding function algebra. However, since $\mathscr{A}$ is also a Hilbert space, we shall use Fourier coefficients instead of Gelfand transforms.

2. Notation and terminology

Let $\mathscr{A}$ be a commutative H^*-algebra [1, 6, 7] such that $\mathscr{A}\neq\{0\}$. Since the case when $\mathscr{A}$ is the space $L^2(G)$ for a compact Abelian group with the usual convolution has particular significance for our work, the multiplication in $\mathscr{A}$ will be denoted by $*$. The fact that the involution on $\mathscr{A}$ is also denoted by $*$ will cause no confusions.

Since by the definition of $\mathscr{A}$, we have $f^* * f\neq 0$ if $0\neq f\in\mathscr{A}$, it is clear that is $\mathscr{A}$ an admissible algebra. Thus, we can consider the multiplier extension

$$\mathfrak{M}=\mathfrak{M}(\mathscr{A})=\mathfrak{M}(\mathscr{A}, \mathscr{A})$$

of $\mathscr{A}$ [11]. To study $\mathfrak{M}$, we need some relevant properties of $\mathscr{A}$.

By the structure theory of H^*-algebras, there exists a complete orthogonal family $\{e_\alpha\}_{\alpha\in\Gamma}$ (where, of course, $e_\alpha \neq e_\beta$ if $\alpha \neq \beta$) of nonzero irreducible self-adjoint idempotents of $\mathscr{A}$. Since $\mathscr{A}$ is also a Hilbert space, we have

$$f = \sum_{\alpha\in\Gamma} \hat{f}(\alpha) e_\alpha$$

for all $f \in \mathscr{A}$, where the numbers

$$\hat{f}(\alpha) = \frac{1}{\|e_\alpha\|^2} \langle f, e_\alpha \rangle$$

are the Fourier coefficients of f with respect to $\{e_\alpha\}_{\alpha\in\Gamma}$. Hence, it follows that

$$f * e_\alpha = \hat{f}(\alpha) e_\alpha$$

for all $f \in \mathscr{A}$ and $\alpha \in \Gamma$. Moreover, it is clear that the linear hull $\mathscr{T}$ of $\{e_\alpha\}_{\alpha\in\Gamma}$ is a dense ideal in $\mathscr{A}$.

Let $\boldsymbol{C}^\Gamma$ be the set of all complex-valued functions defined on Γ, and consider $\boldsymbol{C}^\Gamma$ equipped with the usual pointwise operations, with the natural involution and with the topology induced by the family of seminorms defined by

$$|\lambda|_\alpha = |\lambda_\alpha| \qquad (\lambda = (\lambda_\alpha) \in \boldsymbol{C}^\Gamma).$$

Then the mapping defined on $\mathscr{A}$ by

$$f \longrightarrow \hat{f}$$

is a continuous $*$-isomorphism of $\mathscr{A}$ into $\boldsymbol{C}^\Gamma$ so that its range is the set of all $(\lambda_\alpha) \in \boldsymbol{C}^\Gamma$ such that

$$\sum_{\alpha\in\Gamma} |\lambda_\alpha|^2 \, \|e_\alpha\|^2 < +\infty \, .$$

Finally, we remark that all the infinite sums occuring in this paper are to be understood in the sense of Definition 2 on p. 127 of [3], but we shall use nets instead of filters.

3. A simple characterization of the elements of $\mathfrak{M}$

LEMMA 3.1. *Let $D \subset \mathscr{A}$. Then D is not a divisor of zero in $\mathscr{A}$* [11] *if and only if for every $\alpha \in \Gamma$ there exists $\varphi \in D$ such that $\hat{\varphi}(\alpha) \neq 0$.*

Proof. Assume first that D is not a divisor of zero in $\mathscr{A}$. Let $\alpha \in \Gamma$. Since $0 \neq e_\alpha \in \mathscr{A}$ and D is not a divisor of zero in $\mathscr{A}$, there exists $\varphi \in D$ such that $e_\alpha * \varphi \neq 0$. Hence, since $e_\alpha * \varphi = \hat{\varphi}(\alpha) e_\alpha$, it follows that $\hat{\varphi}(\alpha) \neq 0$.

Assume now that for every $\alpha \in \Gamma$ there exists $\varphi \in D$ such that $\hat{\varphi}(\alpha) \neq 0$. Let $0 \neq f \in \mathscr{A}$. Then, there exists $\alpha \in \Gamma$ such that $\hat{f}(\alpha) \neq 0$.

Moreover, by the assumption, there exists $\varphi \in D$ such that $\hat{\varphi}(\alpha) \neq 0$. Thus, we have $(f*\varphi)^\wedge(\alpha) = \hat{f}(\alpha)\hat{\varphi}(\alpha) \neq 0$. This implies that $f*\varphi \neq 0$.

Remark 3.2. By this lemma, it is clear that $\varphi \in \mathscr{A}$ is not a divisor of zero in $\mathscr{A}$ if and only if $\hat{\varphi}(\alpha) \neq 0$ for all $\alpha \in \Gamma$.

THEOREM 3.3. *Let* $F \in \mathfrak{M}$. *Then* $\mathscr{T} \subset D_F$ *and* $F(\mathscr{T}) \subset \mathscr{T}$.

Proof. Let $\alpha \in \Gamma$. By Lemma 3.1, there exists $\varphi \in D_F$ such that $\hat{\varphi}(\alpha) \neq 0$. Furthermore, by Lemma 1.5 in [11],

$$F(\varphi * e_\alpha) = F(\varphi) * e_\alpha .$$

Thus, we have

$$F(\hat{\varphi}(\alpha) e_\alpha) = F(\varphi)^\wedge(\alpha) e_\alpha .$$

Hence, since F is linear and $\hat{\varphi}(\alpha) \neq 0$, it follows that $e_\alpha \in D_F$ and

$$F(e_\alpha) = \frac{F(\varphi)^\wedge(\alpha)}{\hat{\varphi}(\alpha)} e_\alpha .$$

Finally, again by Lemma 1.5 in [11], the rest of the proof is quite obvious.

Remark 3.4. By this theorem every $F \in \mathfrak{M}$ can be written in the form

$$F = \overline{F|\mathscr{T}}$$

[11]. Hence, it is clear that $\mathfrak{M}$ is isomorphic to the composition algebra of all multipliers from $\mathscr{T}$ into $\mathscr{T}$.

4. Fourier coefficients of the elements of $\mathfrak{M}$

DEFINITION 4.1. For $F \in \mathfrak{M}$, let $\hat{F}$ be the function defined on Γ by

$$\hat{F}(\alpha) = F(e_\alpha)^\wedge(\alpha) .$$

The numbers $\hat{F}(\alpha)$ are called the *Fourier coefficients of* F *with respect to* $\{e_\alpha\}_{\alpha \in \Gamma}$.

Remark 4.2. Observe that this definition may cause no confusion, since we have

$$\hat{F}_f = \hat{f}$$

for all $f \in \mathscr{A}$ [11].

THEOREM 4.3. *Let* $F \in \mathfrak{M}$. *Then*

$$F(\varphi) = \sum_{\alpha \in \Gamma} \hat{F}(\alpha) \hat{\varphi}(\alpha) e_\alpha$$

for all $\varphi \in D_F$.

Proof. Let $\varphi \in D_F$. Then, we have

$$F(\varphi)=\sum_{\alpha\in\Gamma} F(\varphi)^{\wedge}(\alpha)e_\alpha .$$

On the other hand, it is clear that

$$F(\varphi)^{\wedge}(\alpha)=(F(\varphi)*e_\alpha)^{\wedge}(\alpha)=(F(e_\alpha)*\varphi)^{\wedge}(\alpha)=F(e_\alpha)^{\wedge}(\alpha)\hat{\varphi}(\alpha)=\hat{F}(\alpha)\hat{\varphi}(\alpha)$$

for all $\alpha\in\Gamma$.

THEOREM 4.4. *Suppose that* $(\lambda_\alpha)\in \boldsymbol{C}^\Gamma$, *and let* F *be the function defined on*

$$D=\{\varphi\in\mathscr{A}:\ \sum_{\alpha\in\Gamma}|\lambda_\alpha|^2\,|\hat{\varphi}(\alpha)|^2\,\|e_\alpha\|^2<+\infty\}$$

by

$$F(\varphi)=\sum_{\alpha\in\Gamma}\lambda_\alpha\hat{\varphi}(\alpha)e_\alpha .$$

Then $F\in\mathfrak{M}$ *and* $\hat{F}(\alpha)=\lambda_\alpha$ *for all* $\alpha\in\Gamma$.

Proof. Since $\mathscr{T}\subset D$, D is not a divisor of zero in $\mathscr{A}$. Moreover, we have

$$(F(\varphi)*\psi)^{\wedge}(\alpha)=\lambda_\alpha\hat{\varphi}(\alpha)\hat{\psi}(\alpha)=(\varphi*F(\psi))^{\wedge}(\alpha)\qquad(\alpha\in\Gamma),$$

i.e.,

$$F(\varphi)*\psi=\varphi*F(\psi)$$

for all $\varphi,\psi\in D$. Therefore, $F\in\mathscr{M}$ [11].

On the other hand, if $\varphi\in D_{\bar{F}}$, then

$$\lambda_\alpha\hat{\varphi}(\alpha)e_\alpha=\varphi*F(e_\alpha)=\bar{F}(\varphi)*e_\alpha=\bar{F}(\varphi)^{\wedge}(\alpha)e_\alpha ,$$

i.e.,

$$\lambda_\alpha\hat{\varphi}(\alpha)=\bar{F}(\varphi)^{\wedge}(\alpha)$$

for all $\alpha\in\Gamma$. This implies that $\varphi\in D$. Consequently, $F\in\mathfrak{M}$.

Finally, it is clear that

$$\hat{F}(\alpha)=F(e_\alpha)^{\wedge}(\alpha)=(\lambda_\alpha e_\alpha)^{\wedge}(\alpha)=\lambda_\alpha$$

for all $\alpha\in\Gamma$.

THEOREM 4.5. *The mapping defined on* $\mathfrak{M}$ *by*

$$F\longrightarrow\hat{F}$$

is an algebra isomorphism of $\mathfrak{M}$ *onto* $\boldsymbol{C}^\Gamma$.

Proof. By Theorems 4.3 and 4.4, it is clear that this mapping is an injection from $\mathfrak{M}$ onto $\boldsymbol{C}^\Gamma$.

Straightforward calculations show that this mapping preserves the algebra operetions. For example, if $F, G\in\mathfrak{M}$, then

$$(F*G)(e_\alpha)=F(G(e_\alpha))=F(\hat{G}(\alpha)e_\alpha)=\hat{G}(\alpha)F(e_\alpha)=\hat{F}(\alpha)\hat{G}(\alpha)e_\alpha ,$$

and hence

$$(F*G)^\wedge(\alpha)=(F*G)(e_\alpha)^\wedge(\alpha)=\hat{F}(\alpha)\hat{G}(\alpha)$$

for all $\alpha\in\Gamma$.

DEFINITION 4.6. Define the involution on $\mathfrak{M}$ such that the mapping $F\longrightarrow\hat{F}$ be a $*$-isomorphism of $\mathfrak{M}$ onto $\boldsymbol{C}^\Gamma$, i.e., for every $F\in\mathfrak{M}$ denote F^* the unique element of $\mathfrak{M}$ such that

$$(F^*)^\wedge=(\hat{F})^* .$$

Remark 4.7. In general, a conjugate-linear version of Theorem 1 in [12] can be used to extend involution to the multiplier extension.

THEOREM 4.8. *Let* $F\in\mathfrak{M}$. *Then* $D_{F^*}=D_F^*=D_F$ *and*

$$F^*(\varphi)=F(\varphi*)^*$$

for all $\varphi\in D_{F^*}$.

Proof. This follows immediately from Theorems 4.3 and 4.4, since the involution on $\mathscr{A}$ is continuous.

DEFINITION 4.9. Define the topology on $\mathfrak{M}$ such that the mapping $F\longrightarrow\hat{F}$ be also a topological isomorphism of $\mathfrak{M}$ onto $\boldsymbol{C}^\Gamma$, i.e., for example, consider $\mathfrak{M}$ topologized by the family of seminorms $\|\ \|_\alpha$ defined on $\mathfrak{M}$ by

$$\|F\|_\alpha=|\hat{F}|_\alpha=\frac{1}{\|e_\alpha\|}\|F(e_\alpha)\| .$$

Remark 4.10. In general, the Mikusiński-type convergence of sequences or nets [11] can be used to define a reasonable topology on the multiplier extension.

THEOREM 4.11. *Let* $F\in\mathfrak{M}$. *Then*

$$F=\sum_{\alpha\in\Gamma}\hat{F}(\alpha)e_\alpha$$

in $\mathfrak{M}$.

Proof. For each finite subset A of Γ let

$$F_A=\sum_{\alpha\in A}\hat{F}(\alpha)e_\alpha ,$$

and consider the family of all finite subsets of Γ directed by set inclusion. We must show that the net (F_A) converges to F in $\mathfrak{M}$.

If $\alpha\in\Gamma$, then we have

$$\hat{F}_A(\alpha)=\hat{F}(\alpha)$$

for all finite subset A of Γ such that $\alpha\in A$. Hence, it is clear that

$$\lim_A\hat{F}_A(\alpha)=\hat{F}(\alpha)$$

for all $\alpha\in\Gamma$. This implies that

$$\lim_A F_A = F$$

in $\mathfrak{M}$.

Remark 4.12. By this theorem, it is clear that $\mathscr{T}$ is also a dense ideal in $\mathfrak{M}$.

5. Further results on the domains of the elements of $\mathfrak{M}$

THEOREM 5.1. *Suppose that Γ is countable, and let $F \in \mathfrak{M}$. Then there exists $\varphi \in D_F$ such that φ is not a divisor of zero in $\mathscr{A}$.*

Proof. If Γ is finite, then the statement is an immediate consequence of Lemma 3.1 and Theorem 3.3.

Suppose now that Γ is countably infinite, i.e., $\Gamma = \{\alpha_n\}_{n=1}^{\infty}$. Let

$$\varphi = \sum_{n=1}^{\infty} \frac{1}{n(|\hat{F}(\alpha_n)|+1)\,\|e_{\alpha_n}\|}\, e_{\alpha_n} .$$

Then, by Lemma 3.1, φ is not a divisor of zero in $\mathscr{A}$. Moreover, by Theorems 4.3 and 4.4, it is clear that $\varphi \in D_F$.

Remark 5.2. If Γ is countable, then this theorem allows us to write every $F \in \mathfrak{M}$ in the form

$$F = \frac{F(\varphi)}{\varphi}$$

where $\varphi \in D_F$ such that φ is not a divisor of zero in $\mathscr{A}$ [11].

Thus, if Γ is countable, then $\mathfrak{M}$ is isomorphic to the classical quotient algebra of $\mathscr{A}$. If Γ is uncountable, then the classical quotient algebra of $\mathscr{A}$ does not exists, since in this case every element of $\mathscr{A}$ is a divisor of zero in $\mathscr{A}$.

THEOREM 5.3. *Suppose that Γ is not finite, and let $\varphi \in \mathscr{A} \setminus \mathscr{T}$. Then there exists $F \in \mathfrak{M}$ such that $\varphi \notin D_F$.*

Proof. For $\alpha \in \Gamma$, let

$$\lambda_\alpha = \begin{cases} \dfrac{1}{\hat{\varphi}(\alpha)} & \text{if} \quad \hat{\varphi}(\alpha) \neq 0 , \\ 0 & \text{if} \quad \hat{\varphi}(\alpha) = 0 . \end{cases}$$

By Theorem 4.5, there exists $F \in \mathfrak{M}$ such that

$$\hat{F}(\alpha) = \frac{\lambda_\alpha}{\|e_\alpha\|}$$

for all $\alpha \in \Gamma$. Since

$$\sum_{\alpha \in \Gamma} |\hat{F}(\alpha)|^2 \, |\hat{\varphi}(\alpha)|^2 \, \|e_\alpha\|^2 = \sum_{\alpha \in \Gamma} 1 = +\infty ,$$

it is clear that $\varphi \notin D_F$.

THEOREM 5.4. *Let $F \in \mathfrak{M}$. Then the following propositions are pairwise equivalent*:

(i) $D_F = \mathscr{A}$;

(ii) F *is continuous*;

(iii) $\hat{F}$ *is bounded.*

Proof. By Theorem 1.1.1 in [5], it is clear that (i) implies (ii). (For a more general situation see [13].)

Suppose now that F is continuous. Then there exists $0 < M \in \boldsymbol{R}$ such that

$$\|F(\varphi)\| \leqq M \|\varphi\|$$

for all $\varphi \in D_F$. Hence, it follows that

$$|\hat{F}(\alpha)| = \frac{1}{\|e_\alpha\|} \|F(e_\alpha)\| \leqq M$$

for all $\alpha \in \Gamma$, i.e., $\hat{F}$ is bounded.

Finally, to prove that (iii) implies (i), suppose that $\hat{F}$ is bounded. Then, we have

$$\sum_{\alpha \in \Gamma} |\hat{F}(\alpha)|^2 |\hat{\varphi}(\alpha)|^2 \|e_\alpha\|^2 \leqq \sup_{\alpha \in \Gamma} |\hat{F}(\alpha)| \sum_{\alpha \in \Gamma} |\hat{\varphi}(\alpha)|^2 \|e_\alpha\|^2 < +\infty$$

for all $\varphi \in \mathscr{A}$. Hence, by Theorems 4.3 and 4.4, it is clear that $D_F = \mathscr{A}$.

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